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## DETERMINACY WITH COMPLICATED STRATEGIES

ALEXANDER S. KECHRIS

**ABSTRACT.** For any class of functions  $\mathfrak{F}$  from  $\mathbf{R}$  into  $\mathbf{R}$ ,  $\text{AD}(\mathfrak{F})$  is the assertion that in every two person game on integers one of the two players has a winning strategy in the class  $\mathfrak{F}$ . It is shown, in  $ZF + DC + V = L(\mathbf{R})$ , that for any  $\mathfrak{F}$  of cardinality  $\leq 2^{\aleph_0}$  (i.e. any  $\mathfrak{F}$  which is a surjective image of  $\mathbf{R}$ )  $\text{AD}(\mathfrak{F})$  implies  $\text{AD}$  (the Axiom of Determinacy).

**1. Statement of the results.** Let  $\omega = \{0, 1, 2, \dots\}$  be the set of natural numbers and let  $\mathbf{R} = \omega^\omega$  be the set of all infinite sequences from  $\omega$ , called for simplicity *reals* in the following. Given any set  $A \subseteq \mathbf{R} \times \mathbf{R}$  we associate with  $A$  the following infinite game also denoted by  $A$ : In a run of the game players I and II choose alternatively natural numbers  $x(0), y(0), x(1), y(1), \dots$

$$\begin{array}{rcll} \text{I} & x(0) & x(1) & \dots \\ \text{II} & y(0) & y(1) & \dots \end{array}$$

and player I wins this run if  $(x, y) \in A$ ; otherwise II wins. A strategy for player I is a map  $\sigma: \omega^{<\omega} \rightarrow \omega$ , where  $\omega^{<\omega}$  = the set of all finite sequences from  $\omega$ . If II plays  $(y(0), y(1), \dots)$  then I follows the strategy  $\sigma$  if he responds by playing  $x(0), x(1), \dots$ , where  $x(n) = \sigma(y|n)$ , with  $y|n = (y(0), \dots, y(n-1))$ . The strategy  $\sigma$  is winning for player I in the game  $A$ , if for all  $y \in \mathbf{R}$  when II plays  $y$  and I plays  $x$  following  $\sigma$  then  $(x, y) \in A$ . Similarly we define the concept of a strategy  $\tau$  for player II and what it means to be winning for him. The set  $A$  is *determined* if either player I or II has a winning strategy for the game  $A$ . The *Axiom of Determinacy* (AD) is the assertion that all sets  $A \subseteq \mathbf{R} \times \mathbf{R}$  are determined.

If  $\sigma$  is a strategy for player I, then  $\sigma$  can be viewed as a function  $\sigma^*: \mathbf{R} \rightarrow \mathbf{R}$  given by  $\sigma^*(y) = x$ , where  $x(n) = \sigma(y|n)$ . Similarly if  $\tau$  is a strategy for II we denote by  $\tau^*$  the corresponding function. Thus AD can be rewritten as follows:

For all  $A \subseteq \mathbf{R} \times \mathbf{R}$ , either there is a strategy  $\sigma^*$  such that  $\forall y(\sigma^*(y), y) \in A$  or else there is a strategy  $\tau^*$  such that  $\forall x(x, \tau^*(x)) \notin A$ .

The functions of the form  $\sigma^*$  or  $\tau^*$  are clearly continuous functions on  $\mathbf{R}$  of a very special kind, i.e. Lipschitz. What if we consider instead "strategies" which are more complicated? To make this question precise let us give the following definition.

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DEFINITION. Let  $\mathfrak{F}$  be a collection of functions from  $\mathbf{R}$  into  $\mathbf{R}$ . By  $\text{AD}(\mathfrak{F})$  we abbreviate the statement:

For all  $A \subseteq \mathbf{R} \times \mathbf{R}$  either there is  $f \in \mathfrak{F}$  with  $\forall y(f(y), y) \in A$  or there is  $g \in \mathfrak{F}$  with  $\forall x(x, g(x)) \notin A$ .

A geometric reformulation of  $\text{AD}(\mathfrak{F})$  is the following: If  $P \subseteq \mathbf{R} \times \mathbf{R}$  is a relation and  $h: \mathbf{R} \rightarrow \mathbf{R}$  is a function, then we say that  $h$  *uniformizes*  $P$  in the  $y$ -direction if  $\forall x(x, h(x)) \in P$ . We say that  $h$  *uniformizes*  $P$  in the  $x$ -direction if  $\forall y(h(y), y) \in P$ . Then  $\text{AD}(\mathfrak{F})$  asserts that for any  $A \subseteq \mathbf{R} \times \mathbf{R}$  there is a function in  $\mathfrak{F}$  which uniformizes  $A$  in the  $x$ -direction or there is a function in  $\mathfrak{F}$  which uniformizes  $(\mathbf{R} \times \mathbf{R}) - A$  in the  $y$ -direction.

Clearly  $\text{AD}$  implies  $\text{AD}(\mathfrak{F})$  for any  $\mathfrak{F}$  containing the Lipschitz functions. Mycielski [7] has considered first the question of the strength of  $\text{AD}(\text{Continuous})$ . Later H. Friedman [2] asked more generally about  $\text{AD}(\text{Borel})$ ,  $\text{AD}(\text{Projective})$ , etc. A. Blass [1] proved that  $\text{AD}(\text{Continuous}) \Rightarrow \text{AD}$  and Kunen [6] strengthened it substantially by showing that  $\text{AD}(\text{Borel}) \Rightarrow \text{AD}$ , and indeed that  $\text{AD}(\Delta_2^1) \Rightarrow \text{AD}$ . Some further results have been obtained in [3].

We prove in this paper an almost optimal generalization of these results, at least in the case  $V = L(\mathbf{R})$ , where  $L(\mathbf{R})$  is the smallest inner model of  $ZF$  containing  $\mathbf{R}$ . First let us say that a class  $\mathfrak{F}$  of functions from  $\mathbf{R}$  into  $\mathbf{R}$  has *cardinality*  $\leq 2^{\aleph_0}$  if there is a surjection  $p: \mathbf{R} \rightarrow \mathfrak{F}$ . Then we have (assuming  $\mathfrak{F} \supseteq \text{Lipschitz}$ )

THEOREM ( $ZF + DC$ ). (a) For any class of functions  $\mathfrak{F}$  (from  $\mathbf{R}$  into  $\mathbf{R}$ ) which has cardinality  $\leq 2^{\aleph_0}$ ,  $\text{AD}(\mathfrak{F}) \Rightarrow L(\mathbf{R}) \models \text{AD}$ . In particular, if  $V = L(\mathbf{R})$  and  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$ ,  $\text{AD}(\mathfrak{F}) \Rightarrow \text{AD}$ .

(b) If  $\mathfrak{F}$  is a class of functions,  $\mathfrak{F} \subseteq L(\mathbf{R})$  and  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$ ,  $\text{AD}(\mathfrak{F}) \Rightarrow \text{AD}$ .

COROLLARY. The following two theories are equiconsistent:

- (i)  $\text{AD}$ .
- (ii) There is a class  $\mathfrak{F}$  of functions (from  $\mathbf{R}$  into  $\mathbf{R}$ ) of cardinality  $\leq 2^{\aleph_0}$  such that  $\text{AD}(\mathfrak{F})$  holds.

REMARKS. (1) Clearly if  $\mathfrak{F}$  = the class of all continuous,  $\Delta_2^1$ , projective, etc. functions, then  $\mathfrak{F} \subseteq L(\mathbf{R})$  and  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$ , so (b) is a strengthening of the earlier results, and provides a full answer to the question of H. Friedman.

(2) If  $\mathfrak{F}$  = the class of *all* functions on  $\mathbf{R}$ , then the assumption that every  $A \subseteq \mathbf{R} \times \mathbf{R}$  can be uniformized implies trivially  $\text{AD}(\mathfrak{F})$ , thus  $\text{AD}(\mathfrak{F})$  is a weak hypothesis (equiconsistent with  $ZF$ ).

(3) Our method of proof of (b) provides also simple proofs of some earlier results, such as  $\text{AD}(\text{Borel}) \Rightarrow \text{AD}$ . We explain this after giving the proof of the main theorem.

In conclusion, we would like to thank H. Becker for pointing out that the conditions on  $\mathfrak{F}$  under which an original version of (a) was proved amount to just that  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$ .

**2. Proofs.** We first prove part (a) of the theorem. Our argument combines the ideas of [4 and 5], with which we have to assume familiarity. (The proof of (b), which contains the main new idea of this paper, is however self-contained modulo just the statement of part (a).)

Let  $\mathfrak{F}$  be a class of functions of cardinality  $\leq 2^{\aleph_0}$  such that  $\text{AD}(\mathfrak{F})$  holds (we assume of course throughout  $ZF + DC$ ). In order to prove that  $L(\mathbf{R}) \models \text{AD}$  we will use the following result proved in [4]:

If for any  $\lambda < \Theta$  there is a cardinal  $\kappa > \lambda$  with the strong partition property  $\kappa \rightarrow (\kappa)^\kappa$ , then  $L(\mathbf{R}) \models \text{AD}$ .

So it is enough to prove that  $\forall \lambda < \Theta \exists \kappa > \lambda (\kappa \rightarrow (\kappa)^\kappa)$ . Now in [5] it is indeed shown that

$$\text{AD} \Rightarrow \forall \lambda < \Theta \exists \kappa > \lambda (\kappa \rightarrow (\kappa)^\kappa).$$

Our main observation is that one can still prove

$$\text{AD}(\mathfrak{F}) \Rightarrow \forall \lambda < \Theta \exists \kappa > \lambda (\kappa \rightarrow (\kappa)^\kappa),$$

provided  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$ .

Indeed fix  $\lambda < \Theta$  and let  $\pi: \mathbf{R} \rightarrow \mathfrak{F}$  be a surjection. Let  $A \subseteq \mathbf{R}^3$  be the relation that codes  $\pi$ , i.e.

$$(x, y, z) \in A \Leftrightarrow \pi(x)(y) = z.$$

Let now, using the terminology of [5],  $\Gamma$  be a Spector pointclass closed under  ${}^3E$  such that  $\lambda < \kappa = 0(\Delta)$  ( $=$  supremum of the ranks of prewellorderings in  $\Delta$ ) and  $A \in \Delta$ . Now by [5, Theorem 1.1], if we knew that  $\text{AD}$  holds then we would have that  $\kappa$  has the strong partition property. The following change in the proof of [5, Theorem 1.1] makes the same proof work under the weaker assumption that  $\text{AD}(\mathfrak{F})$  holds:

In p. 80 of [5] modify the definition of  $\Sigma_1^1(\chi)$  formulas to the obvious definition of  $\Sigma_1^1(\chi, \psi)$  formulas, where  $\chi$  denotes now a partial function from  $\mathbf{R} \times \mathbf{R}$  into  $\omega$  and  $\psi$  a partial function from  $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$  into  $\omega$ . Then let  $\Sigma_1^1(\chi; A)$  be  $\Sigma_1^1(\chi; \psi_A)$ , where  $\psi_A$  is the characteristic function of  $A$ . Replace now in the rest of the proof of [5, Theorem 1.1],  $\Sigma_1^1(\chi)$  by  $\Sigma_1^1(\chi; A)$ .

We now come to the proof of (b). So assume  $\mathfrak{F} \subseteq L(\mathbf{R})$ ,  $\mathfrak{F}$  has cardinality  $\leq 2^{\aleph_0}$  and  $\text{AD}(\mathfrak{F})$  holds. Then by (a) we surely have that at least  $\text{AD}^{L(\mathbf{R})}$  holds. In order to prove full  $\text{AD}$ , consider an arbitrary game  $A \subseteq \mathbf{R} \times \mathbf{R}$ . Define then the following auxiliary game  $A^* \subseteq \mathbf{R} \times \mathbf{R}$ :

$$A^*(\sigma, \tau) \Leftrightarrow \sigma \text{ a strategy for I } \wedge [\tau \text{ is a strategy for II } \Rightarrow \sigma * \tau \in A],$$

where for strategies  $\sigma, \tau$  for I, II, respectively, we let  $\sigma * \tau$  be the outcome of the run of the game in which I follows  $\sigma$  and II follows  $\tau$ . Thus  $A^*$  is almost like  $A$  except that players I and II play against each other's strategies instead of individual moves. (Strategies are of course viewed as reals.)

Now by our hypothesis there is  $f \in \mathfrak{F}$  such that  $\forall \tau (f(\tau), \tau) \in A^*$  or else there is  $g \in \mathfrak{F}$  with  $\forall \sigma (g(\sigma), \sigma) \notin A^*$ . The two cases are similar, so let us assume that we have the first one. Put

$$B = \{f(\tau) * \tau : \tau \text{ is strategy for II}\}.$$

Since  $\forall \tau (f(\tau), \tau) \in A^*$ , if  $\tau$  is a strategy for II, then  $f(\tau) * \tau \in A$ , so  $B \subseteq A$ . But  $f \in \mathfrak{F} \subseteq L(\mathbf{R})$ , thus  $B \in L(\mathbf{R})$ , and so by  $\text{AD}^{L(\mathbf{R})}$ ,  $B$  is determined. If I has a winning strategy in  $B$  then I clearly has a winning strategy in  $A$  and we are done; else II has a winning strategy  $\tau_0$  in  $B$ . Let  $\sigma_0 = f(\tau_0)$ . Then  $\sigma_0 * \tau_0 \notin B$ , but by definition of  $B$  we also have  $\sigma_0 * \tau_0 \in B$ , a contradiction, and our proof is complete.

As a final illustration of this technique let us give a simple proof of Kunen's result that  $\text{AD}(\text{Borel}) \Rightarrow \text{AD}$ .

Assume  $\text{AD}(\text{Borel})$ . Then it is easy to verify that we have hyperdegree determinacy, i.e., every set of hyperdegrees contains or is disjoint from a cone of hyperdegrees. From this it follows as usual that  $\aleph_1$  is measurable and thus  $\Sigma_1^1$ -Determinacy holds (the arguments involved here are of course due to Martin for the case of Turing degrees, and carry over trivially to hyperdegrees). Let now  $A \subseteq \mathbf{R} \times \mathbf{R}$  be any game and consider the game  $A^*$  as before. Using  $\text{AD}(\text{Borel})$ , let us say that  $f$  is Borel with  $\forall \tau (f(\tau), \tau) \in A^*$ . Form as before the set  $B = \{f(\tau) * \tau : \tau \text{ is a strategy for II}\}$ . Then  $B \in \Sigma_1^1$ , so  $B$  is determined and the rest of the argument is exactly the same.

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